The Q-curvature on a 4-dimensional Riemannian manifold (M, g) with $\int_M Q dV_g = 8\pi^2$

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1 Introduction

One of the most important problem in conformal geometry is the construction of conformal metrics for which a certain curvature quantity equals a prescribed function, e.g. a constant. In two dimensions, the problem of prescribed Gaussian curvature asks the following: given a smooth function K on (M, g_0), can we find a metric g conformal to g_0 such that K is the Gaussian curvature of the new metric g? If let $g = e^{2u} g_0$ for some $u \in C^\infty(M)$, then the problem is equivalent to solving the nonlinear elliptic equation:

$$\Delta u + K e^{2u} - K_0 = 0, \tag{1.1}$$

where Δ denotes the Beltrami-Laplacian of (M, g_0) and K_0 is the Gaussian curvature of g_0.

In dimension four, there is an analogous formulation of equation (1.1). Let (M, g) be a compact Riemannian four manifold, and let Ric and R denote respectively the Ricci tensor and the scalar curvature of g. A natural conformal invariant in dimension four is

$$Q = Q_g = -\frac{1}{12}(\Delta R - R^2 + 3|Ric|^2).$$

Note that, under a conformal change of the metric

$$\tilde{g} = e^{2u} g,$$

the quantity Q transforms according to

$$2Q_{\tilde{g}} = e^{-4u}(Pu + 2Q_g), \tag{1.2}$$

where $P = P_g$ denotes the Paneitz operator with respect to g, introduced in [P]. For any g the operator P_g acts on a smooth function u on M via

$$P_g(u) = \Delta_g^2 u + div(\frac{2}{3}R_g - 2Ric_g)du,$$

which plays a similar role as the Laplace operator in dimension two. Note that the Paneitz operator is conformal invariant in the sense that

$$P_{\tilde{g}} = e^{-4u} P_g$$

for any conformal metric $\tilde{g} = e^{2u} g$.

It follows that the expression $k = k_g := \int_M Q dV_g$ is conformally invariant. Moreover, in view of relation (1.2), a natural problem to propose is to prescribe the Q-curvature: that is, to ask whether on a given four-manifold (M, g) there exists a conformal metric $\tilde{g} := e^{2u} g$ for which the Q-curvature of $\tilde{g}$ equlas the prescribed function $\tilde{Q}$? This is related to solving the following equation

$$P_g u + 2Q_g = 2\tilde{Q}e^{4u}. \tag{1.3}$$

This equation is the Euler-Language equation of the functional

$$II_g(u) = \int_M u P_g u dV_g + 4\int_M Q_g u dV_g - (\int_M Q_g dV_g) \log \int_M \tilde{Q}e^{4u} dV_g. \tag{1.4}$$

A partial affirmative answer to the problem (1.3) in the case that $\tilde{Q}$ equals some constant is given by Chang-Yang [C-Y] provided that the Paneitz operator is weakly positive and the integral k is less than $8\pi^2$. In view of a result of Gursky [G] the former hypothesis is satisfied whenever $k > 0$ and provided (M, g) is of positive Yamabe type. The result of Chang-Yang has been extended recently by Djadli-Malchiodi [D-M] to the case in which P_g has no kernel and k is not positive integer multiple of $8\pi^2$.

In the critical case, when $k = 8\pi^2$, the study of equation (1.3) becomes rather delicate. In this case the functional II_g fails to satisfy standard compactness conditions like the Palais-Smale condition, and generally blow-up may occur. Note that when $(M, g) = (S^4, g_c)$, the above equation (1.3) is reduced to the following one

$$P_g u + 6 = 2\tilde{Q}e^{4u}. \tag{1.5}$$

This is the analogue of the well-known Nirenberg's problem. This problem has been recently studied by many authors (please see [W-X], [M-St] and the reference there in). We remark that, similar to Nirenberg's problem, there are some obstructions to the existence of solution to equation (1.5) in the standard four-sphere case. The Gauss-Bonnet-Chern formula implies that there could not be a solution if $\tilde{Q} \leq 0$. On the other hand, one has the identities of Kazdan-Warner type to this equation.

The main goal of this paper is to study the equation (1.3) with critical value $k = 8\pi^2$. We shall pursue a variational approach which was used in [D-J-L-W]. Let (M, g) be any closed four dimensional Riemannian manifold with positive P_g, i.e., $\int_M u P_g u dV_g \geq 0$ and $ker P_g = \{constants\}$. Then we have

$$\int_M u P_g u dV_g \geq \lambda \int_M |\nabla_g u|^2 dV_g$$

for some positive λ and the following improved Adams-Fontana inequality [C-Y]:

$$\log \int_M e^{4u} dV_g \leq \frac{1}{8\pi^2} \int_M u P_g u dV_g + 4\int_M u dV_g + C, \ \forall u \in W^{2,2}(M). \tag{1.6}$$

We consider (for any small $\epsilon > 0$)

$$II_\epsilon(u) = \int_M \langle u, u\rangle dV_g + 4(1 - \frac{\epsilon}{8\pi^2}) \int_M Q_g u dV_g - (8\pi^2 - \epsilon) \log \int_M \tilde{Q}e^{4u} dV_g,$$

where we denote

$$\langle u, v\rangle = \Delta_g u \Delta_g v + (\frac{2}{3}R_g(\nabla u, \nabla v) - 2Ric_g(\nabla u, \nabla v)).$$

By using the inequality (1.6), it is not so difficult to prove that

$$\inf II_\epsilon(u) > -\infty, \forall \epsilon > 0, \text{ and moreover, } II_\epsilon \text{ has a minimal point } u_\epsilon.$$

For this minimizing sequence u_ϵ, two possibilities may occur: let $m_\epsilon = u_\epsilon(x_\epsilon) = \max_{x\in M} u_\epsilon(x)$,

(1) $\sup_\epsilon m_\epsilon < +\infty$, then, by passing to a subsequence, $\{u_\epsilon\}$ converges to some u_0 as $\epsilon \to 0$, and u_0 minimizes II.

(2) $m_\epsilon \to +\infty$, as $\epsilon \to 0$. We call, in this case, the u_ϵ blows up.

One of the main concern is to prove that, if the second case happens, then we find an explicit bound for the II_ϵ. More precisely, we have

$$\inf_{u\in W^{2,2}(M)} II(u) \geq \Lambda_g(\tilde{Q}, p), \tag{1.7}$$

where

$$\Lambda_g(\tilde{Q}, p) = -16\pi^2 \log \frac{\sqrt{3\tilde{Q}(p)}}{12} - 8\pi^2 \log 8\pi^2 - 16\pi^2 S_0(p) + 2\int_M QG_p dV_g + (8/3 - 16)\pi^2,$$

p is the bubble point, and $S_0(p)$ is the constant term of the Green function at point p (please see section 6).

On the other hand , if we can construct some test function sequence ϕ_ϵ, s.t.

$$II(\phi_\epsilon) < \Lambda_g(\tilde{Q}, p),$$

we see that the blow-up does not happen. Therefore, we can get some sufficient condition under which (1.3) has a solution.

One of our main theorem in this paper is as follows.

Theorem 1.1. *Let (M, g) be a closed Riemannian manifold of dimension four, with $k = 8\pi^2$. Suppose P_g is positive. If the $\inf_{u\in W^{2,2}(M)} II(u)$ can not be attained, i.e. equation (1.3) has no minimal solution, then*

$$\inf_{u\in W^{2,2}(M)} II(u) = \inf_{p\in M} \Lambda_g(\tilde{Q}, p). \tag{1.8}$$

Now let p' be a point s.t.

$$\Lambda_g(\tilde{Q}, p') = \inf_{x\in M} \Lambda_g(\tilde{Q}, x),$$

we will prove that p' is in fact determined by the conformal class $[g]$ of (M, g).

Another main result in this paper is the existence theorem of the equation (1.3).

Theorem 1.2. *Let (M,g) be a closed Riemannian manifold of dimension four, with $k=8\pi^2$. Suppose P_g is positive. Let $\tilde{Q}$ be a positive smooth function on M. Assume that $\Lambda_g(\tilde{Q},x)$ achieves its minimum at the point p'. If*

$$\tilde{Q}(p')(\Delta_g S(p')+4|\nabla_g S(p')|^2-\frac{R(p')}{18})+[(2\nabla_g S\nabla_g\tilde{Q})(p')+\frac{1}{4}\Delta_g\tilde{Q}(p')]>0,$$

then equation (1.3) has a minimal solution.

Corollary 1.3. *With the assumption as in Theorem 1.2. If*

$$\Delta_g S(p')+4|\nabla_g S(p')|^2-\frac{R(p')}{18}>0,$$

then M has a constant Q-curvature up to conformal transformations.

It is interesting to note that, in four-dimensional case, the method in [D-J-L-W] can not be directly used. In our case there are some interesting points happens, one is that we use the method [M-2] to collect the nice information around the bubble points. The second one is a new technique used in the derivation of (1.8), where the key point is to calculate

$$\int_{B_\delta\setminus B_{Lr_\epsilon}(x_\epsilon)}|\Delta_g u_\epsilon|^2 dV_g. \tag{1.9}$$

Since the equation (1.3) does not satisfy the Maximal Principle, the method used in [D-J-L-W] does not work here. We will apply the capacity to get the lower bound of (1.9). The usefulness of capacity in similar problems was first discovered by the second author, and has been used in [Li] and [Li-Li].

We remark that the methods in this paper also work for the equation

$$P_g u+16\pi^2=2he^{4u}, \tag{1.10}$$

on any 4-dimensional manifold under the assumptions that P_g is positive and $Vol=1$. Therefore Theorem 1.1 and Theorem 1.2 hold for equation (1.10) (just change $\tilde{Q}$ to h).

2 Preliminary estimate

In this section we collect some useful preliminary facts and then drive some estimates for the solutions. We start with the following lemma.

Lemma 2.1. *For any $\epsilon>0$, II_ϵ has a minimal point.*

Proof. By using the inequality (1.6), it is easy to see that, when $\int_M u dV_g=0$, we have

$$\begin{aligned}
II_\epsilon(u) &= \int_M uP_g u dV_g+4(1-\frac{\epsilon}{8\pi^2})\int_M Q u dV_g-(8\pi^2-\epsilon)\log\int_M\tilde{Q}e^{4u}dV_g\\
&\geq C+\tfrac{\epsilon}{8\pi^2}\int_M uP_g u dV_g+4(1-\frac{\epsilon}{8\pi^2})\int_M Q u dV_g\\
&\geq C+\lambda\tfrac{\epsilon}{8\pi^2}\int_M|\nabla_g u|^2 dV_g+4(1-\frac{\epsilon}{8\pi^2})\int_M Q u dV_g.
\end{aligned}$$

For any $\epsilon_1 > 0$, we have

$$\int_M QudV_g \leq \epsilon_1 \int_M |u|^2 + C_\epsilon \leq \lambda_0 \epsilon_1 \int_M |\nabla u|^2 dV_g + C_\epsilon,$$

where λ_0 is the first eigenvalue of Δ. Then,

$$\int_M |\nabla_g u|^2 dV_g \leq C(\epsilon) II_\epsilon(u) + C \tag{2.1}$$

and then

$$\int_M |\Delta_g u|^2 dV_g \leq \frac{8\pi}{\epsilon} II_\epsilon(u) + C. \tag{2.2}$$

Let $u_k = u_{\epsilon,k}$ be a minimizing sequence of II_ϵ, i.e.

$$II_\epsilon(u_k) \to \inf II_\epsilon(u) = A,$$

which, together with the above inequality, implies that

$$\int_M |\Delta_g u_k|^2 dV_g \leq C,$$

for some constant C which may depend on ϵ. Therefore, by passing to a subsequence, we have $u_k \rightharpoonup u_\epsilon$ and

$$\int_M |\Delta_g u_k|^2 dV_g \to B.$$

Since the functional II_ϵ is invariant under a translation by a constant, we may assume that $\int_M u_k dV_g = 0$, then by (1.6), we can see that $e^{4u_k} \in L^p$ for any $p > 0$.

Set

$$II_\epsilon(u_k) := \int_M |\Delta_g u_k|^2 dV_g + \int_M F(u_k) dV_g,$$

then we have,

$$\lim_{k\to+\infty} \int_M F(u_k) dV_g = A - B, \quad and \quad \lim_{k\to+\infty, m\to+\infty} \int_M F(\frac{u_k + u_m}{2}) dV_g = A - B.$$

Since $II_\epsilon(\frac{u_k+u_m}{2}) \geq A$, we have

$$\frac{1}{4}\int_M (|\Delta_g u_k|^2 + |\Delta_g u_m|^2) dV_g + \frac{1}{2}\int_M \Delta_g u_k \Delta_g u_m dV_g \geq B.$$

Hence

$$\lim_{k\to+\infty, m\to+\infty} \int_M \Delta_g u_k \Delta_g u_m dV_g \geq B.$$

Then

$$\lim_{k\to+\infty, m\to+\infty} \int_M |\Delta_g(u_k - u_m)|^2 dV_g =$$
$$\lim_{k\to+\infty, m\to+\infty} (\int_M |\Delta_g u_k|^2 dV_g + \int_M |\Delta_g u_m|^2 dV_g - 2\int_M \Delta_g u_k \Delta_g u_m dV_g) \leq 0.$$

Therefore, $\{u_k\}$ is a Cauchy sequence in $W^{2,2}(M)$.

□

Lemma 2.2. *We have* $\inf II_\epsilon$ *is decreasing in* ϵ*. Moreover,*

$$\lim_{\epsilon\to 0} II_\epsilon = \inf II.$$

Proof. Since $II_\epsilon(u+c) = II_\epsilon(u)$, we can assume that $\int_M Q_g u dV_g = 0$. Therefore

$$II_{\epsilon'}(u) = II_\epsilon(u) + (\epsilon - \epsilon')\int_M \tilde{Q}e^{4u}.$$

Hence, $\inf II_\epsilon$ is decreasing in ϵ, and $\inf II \le \inf II_\epsilon$.

Let $\epsilon' = 0$, and $II(u_\epsilon) = \inf II_\epsilon(u)$. We have

$$II(u) \ge II_\epsilon(u_\epsilon) - \epsilon\int_M \tilde{Q}e^{4u} dV_g.$$

Letting $\epsilon \to 0$, we get that $\inf II \ge \lim_{\epsilon\to 0} \inf II_\epsilon$.

□

Now let u_ϵ be the minimal point of II_ϵ, it is clear that u_ϵ satisfies the following equation:

$$\begin{cases} P_g u_\epsilon + 2(1 - \frac{\epsilon}{8\pi^2})Q_g = 2(1 - \frac{\epsilon}{8\pi^2})\tilde{Q}e^{4u_\epsilon} \\ \int_M \tilde{Q}e^{4u_\epsilon} dV_g = 8\pi^2. \end{cases}$$

The same proof of Lemma 2.3 in [M-2] yields the following

Lemma 2.3. *There are constants* $C_1(q)$*,* $C_2(q)$*,* $C_3(q)$ *depending only on* p *and* M *such that, for* r *sufficiently small and for any* $x \in M$ *there holds*

$$\int_{B_r(x)} |\nabla^3 u_\epsilon|^q dV_g \le C_1(q) r^{4-3q}, \quad \int_{B_r(x)} |\nabla^2 u_\epsilon|^q dV_g \le C_2(q) r^{4-2q},$$

and

$$\int_{B_r(x)} |\nabla u_\epsilon|^q dV_g \le C_3(q) r^{4-q}$$

where, respectively, $q < \frac{4}{3}$*,* $q < 2$*, and* $q < 4$*.*

3 The proof of Theorem 1.1

Let x_ϵ be the maximum point of u_ϵ. Assume $m_\epsilon = u_\epsilon(x_\epsilon)$, $r_\epsilon = e^{-m_\epsilon}$, and $x_\epsilon \to p$. Let $\{e_i(x)\}$ be an orthogonal basis of TM near p and $exp_x : T_xM \to M$ be the exponential mapping. The smooth mapping $E : B_\delta(p) \times B_r \to M$ is defined as follows,

$$E(x, y) = exp_x(y^i e_i(x)),$$

where B_r is a small ball in $\mathbb{R}^n$. Note that $E(x, \cdot) : T_xM \to M$ are all differential homeomorphism if r is sufficiently small.

We set

$$g_{ij}(x, y) = \langle (exp_x)_* \frac{\partial}{\partial y^i}, (exp_x)_* \frac{\partial}{\partial y^j} \rangle_{E(x,y)}.$$

It is well-known that $g = (g_{ij})$ is smooth, and $g(x, y) = I + O(|y|^2)$ for any fixed x. That is, we are able to find a constant K, s.t.

$$\|g(x, y) - I\|_{C^0(B_\delta(p)\times B_r)} \leq K|y|^2$$

when δ and r are sufficiently small. Moreover, for any $\varphi \in C^\infty(B_\rho(x_k))$ we have

$$\Delta_g u_\epsilon = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^k}(\sqrt{|g|}g^{km}\frac{\partial u_\epsilon(E(x_\epsilon, x))}{\partial x^m}), \quad |\nabla u_\epsilon|^2 = g^{pq}\frac{\partial u_\epsilon(E(x_\epsilon, x))}{\partial x^p}\frac{\partial u_\epsilon(E(x_\epsilon, x))}{\partial x^q},$$

and

$$\int_{B_\delta(x_k)} \varphi dV_g = \int_{E^{-1}(x_k,y)B_{\delta(x_k)}} \varphi(E^{-1}(x_k, y))\sqrt{|g|}dy.$$

We define

$$\tilde{u}_\epsilon(x) = u_\epsilon(E(x_\epsilon, x)),$$

and

$$v_\epsilon(x) = \tilde{u}_\epsilon(r_\epsilon x), v'_\epsilon = v_\epsilon - m_\epsilon.$$

Now v_ϵ, v'_ϵ are functions defined on $B_{\frac{r}{2r_\epsilon}} \subset \mathbb{R}^n$.

We have

$$\Delta^2_{g_\epsilon} v'_\epsilon = r^2_\epsilon O(|\nabla^2 v'_\epsilon|) + r^3_\epsilon O(\nabla v'_\epsilon) + \tilde{Q}_g(E(x_\epsilon, r_\epsilon x))e^{4v'_\epsilon}. \tag{3.1}$$

It follows from Lemma 2.3 that,

$$\|\nabla^2 v'_\epsilon\|_{L^q(B_L)} \leq C(L, q) \text{ and } \|\nabla v'_\epsilon\|_{L^q(B_L)} \leq C'(L, q) \text{ for any } q \in (1, 2).$$

Then (3.1) implies that

$$\|\Delta_{g_\epsilon}(\Delta_{g_\epsilon} v'_\epsilon)\|_{L^q(B_L)} \leq C'(L).$$

Using the standard elliptic estimate, we get

$$\|\Delta_{g_k} v'_\epsilon\|_{W^{2,q}(B_L)} \leq C_2(L).$$

The Sobolev inequality then yields that,

$$\|\Delta_{g_\epsilon} v'_\epsilon\|_{L^q(B_L)} \leq C_3(q, L) \quad for \quad any \quad q \in (0, 4).$$

We therefore have

$$\|v'_\epsilon\|_{W^{2,q}(B_L)} \leq C_4(L).$$

Hence, by using the standard elliptic estimates, we see that v'_ϵ converge smoothly to w, which satisfies

$$\Delta^2_0 w = 2\tilde{Q}(p)e^{4w}.$$

Moreover, it is easy to check that

$$\int_{B_L} \tilde{Q}(p)e^{4w}dx \leq 8\pi^2$$

for any $L > 0$. By the result of [Lin], we have

a) $w = -\log(1 + \frac{\sqrt{3\tilde{Q}(p)}}{12}|x|^2)$, with

$$\tilde{Q}(p)\int_{\mathbb{R}^4} e^{4w} dV_g = 8\pi^2,$$

or

b) w has the following asymptotic behavior:

$$-\Delta w \to a > 0 \text{ as } |x| \to +\infty.$$

We claim that b) does not happen. If it does, then we have

$$\lim_{\epsilon\to+0}\int_{B_R} -\Delta_g v_\epsilon \sim \frac{\omega_3}{4} a R^4.$$

However, it follows from Lemma 2.3 that

$$\int_{B_R} |\Delta_{g_\epsilon} v'_\epsilon| dV_g \le CR^2.$$

This shows the case b) does not happen.

For simplicity, let $\lambda = \frac{\sqrt{3Q(p)}}{12}$, so that we have

$$w = -\log(1 + \lambda|x|^2).$$

Now, we consider the convergence of u_ϵ outside the bubble. By Lemma 2.3, u_ϵ is bounded in $W^{3,q}$ for any $q < \frac{4}{3}$. Then, it is easy to check that $u_\epsilon - \bar{u}_\epsilon \rightharpoonup G_p$, where

$$P_g G_p + 2Q_g = 16\pi^2 \delta_p, \quad \int_M G_p dV_g = 0.$$

To prove the strong convergence of $u_\epsilon - \bar{u}_\epsilon$, we first show the following lemma.

Lemma 3.1. *Given $\Omega \subset\subset M \setminus \{p\}$, there holds*

$$\int_\Omega e^{q(u_\epsilon - \bar{u}_\epsilon)} dV_g < C(\Omega, q)$$

for any $q > 0$.

Proof. Let $f_\epsilon = \tilde{Q}_g e^{4u_\epsilon}$. For any $x \in \Omega$, we have the following representation formula,

$$u_\epsilon(x) - \bar{u}_\epsilon = -\int_M G(x,y) Q_g dV_{g,y} + \int_M G(x,y) f_\epsilon.$$

Hence, if let $\Omega_\epsilon = M \setminus B_{L\epsilon}(x_\epsilon)$, and $\mu_\epsilon = 1/\int_{\Omega_\epsilon} |f| dV_g$, we have, for any $q' > 0$,

$$e^{q'\mu_\epsilon(u_\epsilon - \bar{u}_\epsilon + \int_M G(x,y)Q_g dV_g)} = e^{\int_{\Omega_\epsilon} q' G(x,y)\mu_\epsilon f_\epsilon(y) dV_{g,y} + \int_{B_{Lr_\epsilon}} q' G(x,y)\mu_\epsilon f_\epsilon(y) dV_{g,y}}.$$

Notice that for any $x \in \Omega$, we have

$$\int_{B_{Lr_\epsilon}(x_\epsilon)} q'|G(x,y)|\mu_\epsilon f_\epsilon(y) dV_{g,y} \le C_1(L)\int_{B_{Lr_\epsilon}(x_\epsilon)} f_\epsilon(y) dV_g \le C_2(L),$$

and

$$e^{\int_{\Omega_\epsilon} q'G(x,y)\mu_\epsilon f_\epsilon(y)dV_{g,y}} \le \int_{\Omega_\epsilon} \frac{f_\epsilon(y)}{\|f_\epsilon\|_{L^1(\Omega_\epsilon)}} e^{q'G(x,y)} dV_{g,y}.$$

Therefore, by using the Jensen's inequality and the Fubini's theorem, we obtain

$$\begin{aligned}\int_\Omega e^{\int_{\Omega_\epsilon} q'G(x,y)\mu_\epsilon f_\epsilon(y)dV_{g,y}} dV_g &\le \int_\Omega \frac{f_\epsilon(y)}{\|f_\epsilon\|_{L^1(\Omega_\epsilon)}} \Big(\int_{\Omega_\epsilon} e^{q'G(x,y)} dV_{g,x}\Big) dV_{g,y} \\ &\le C\int_\Omega \frac{f_\epsilon(y)}{\|f_\epsilon\|_{L^1(\Omega_\epsilon)}} \Big(\int_{\Omega_\epsilon} \frac{1}{|x-y|^{\frac{q'}{8\pi^2}}} dV_{g,x}\Big) dV_{g,y}.\end{aligned}$$

The last integral is finite provided $q' < 32\pi^2$. Hence, for any $q > 0$, if ϵ is sufficiently small so that $q \le q'\mu_\epsilon$ we have

$$\int_\Omega e^{q(u_\epsilon(x)-\bar{u}_\epsilon)} dx \le \int_\Omega e^{q'\mu_\epsilon(u_\epsilon(x)-\bar{u}_\epsilon)} dx \le C\int_\Omega e^{\int_{\Omega_\epsilon} q'G(x,y)\mu_\epsilon f_\epsilon(y)dV_{g,y}} dV_g \le C.$$

□

As a consequence of the above lemma, we have

Lemma 3.2. *Let $\Omega \subset\subset M \setminus \{x_0\}$. Then $u_\epsilon - \bar{u}_\epsilon$ converges to G_{x_0} in $C^k(\Omega)$ as $\epsilon \to 0$.*

Proof. It is easy to see that $\bar{u}_\epsilon < C$. Then the lemma follows.

□

Remark: In B_{δ_0}, we set $p = y_\epsilon$ for any ϵ. Clearly, $y_\epsilon \to 0$. Then we also have $u_\epsilon(E(p,x)) - \bar{u}_\epsilon \to G_p(E(p,x))$. Moreover, we may write

$$G(E(p,x)) = -2\log|x| + S_0(p) + S_1(x),$$

where $S_0(p)$ is a constant and $S_1 = O(r^{2+\alpha})$. It is easy to check $\tilde{u}_\epsilon - \bar{u}_\epsilon \to G(E(p,x))$ smoothly in $B_{\delta_0} \setminus B_\delta$ for any fixed δ.

Now, we estimate the lower bound of $\lim_{\epsilon\to 0}\int_M \langle u_\epsilon, u_\epsilon\rangle dV_g$. We write

$$\int_M \langle u_\epsilon, u_\epsilon\rangle dV_g = I_1 + I_2 + I_3,$$

where I_1, I_2, I_3 denote the integrals on $M \setminus B_\delta(x_\epsilon)$, $B_{Lr_\epsilon}(x_\epsilon)$ and $B_\delta \setminus B_{Lr_\epsilon}(x_\epsilon)$ (any fixed L and δ) respectively. We remark that the integral I_1, I_2 can be easily treated due to the above lemmas. On the other hand, by Lemma 2.3, we have

$$\int_{B_\delta\setminus B_{Lr_\epsilon}(x_\epsilon)} |\nabla_g u_\epsilon|^2 dV_g \to \int_{B_\delta(p)} |\nabla_g G|^2 = O(\delta^2).$$

So, the key point is to calculate

$$\int_{B_\delta(x_\epsilon)\setminus B_{Lr_\epsilon}(x_\epsilon)} |\Delta_g u_\epsilon|^2 dV_g.$$

We are going to prove the following lemma.

Lemma 3.3. *We have*

$$\int_{B_\delta(x_\epsilon)\setminus B_{Lr_\epsilon}(x_\epsilon)} |\Delta_g u_\epsilon|^2 dV_g \geq \int_{B_\delta\setminus B_{Lr_\epsilon}} |(1-B|x|^2)\Delta_0 \tilde{u}_\epsilon|^2 dx + J(L,\epsilon,\delta),$$

for some $B>0$, where

$$\lim_{\delta\to 0}\lim_{\epsilon\to 0} J(L,\epsilon,\delta) = 0.$$

Proof. Since we have

$$\begin{aligned}|\Delta_g u_\epsilon|^2 &= |g^{km}\tfrac{\partial^2 \tilde{u}_\epsilon}{\partial x^k \partial x^m} + O(|\nabla \tilde{u}_\epsilon|^2)|^2 \\ &= |g^{km}\tfrac{\partial^2 \tilde{u}_\epsilon}{\partial x^k \partial x^m}|^2 + O(|\nabla^2 \tilde{u}_\epsilon|(|\nabla \tilde{u}_\epsilon|)) + O((|\nabla \tilde{u}_\epsilon|^2)),\end{aligned}$$

and since $\tilde{u}_\epsilon - \bar{u}_\epsilon$ converges to $G_p(E(p,x))$ in $W^{3,q}$ for any $q<\frac{4}{3}$, we get

$$\begin{aligned}&\textstyle\int_{B_\delta\setminus B_{Lr_\epsilon}} O(|\nabla^2 \tilde{u}_\epsilon|(|\nabla \tilde{u}_\epsilon|) + O(|\nabla \tilde{u}_\epsilon|^2) \\ &\qquad \leq C(\|\nabla^2 G_p\|_{L^q(B_\delta\setminus B_{Lr_\epsilon})}\|\nabla_g G_p\|_{L^{q'}}(B_\delta\setminus B_{Lr_\epsilon}) + \|G_p\|_{W^{1,2}(B_\delta\setminus B_{Lr_\epsilon})}) \\ &\qquad = J(L,\epsilon,\delta),\end{aligned}$$

where $\frac{3}{2}<q<2$, and $\frac{1}{q'}+\frac{1}{q}=1$.

Let $g^{km}=\delta^{km}+A^{km}$, with $|A^{km}|\leq K|x|^2$ for any ϵ,k,m. Consequently we have

$$|g^{km}\frac{\partial^2 \tilde{u}_\epsilon}{\partial x^k \partial x^m}|^2 = |\Delta_0 \tilde{u}_\epsilon|^2 + 2\sum_{s,t} A^{st}\Delta_0 \tilde{u}_\epsilon \frac{\partial^2 \tilde{u}_\epsilon}{\partial x^s \partial x^t} + \sum_{k,m,s,t} A^{km}A^{st}\frac{\partial^2 \tilde{u}_\epsilon}{\partial x^k \partial x^m}\frac{\partial^2 \tilde{u}_\epsilon}{\partial x^s \partial x^t}.$$

It is clear that

$$2\int_{B_\delta\setminus B_{Lr_\epsilon}} |A^{st}\Delta_0 \tilde{u}_\epsilon \frac{\partial^2 \tilde{u}_\epsilon}{\partial x^s \partial x^t}| \leq K\int_{B_\delta\setminus B_{Lr_\epsilon}} (|x|^2|\Delta_0 \tilde{u}_\epsilon|^2 + |x|^2|\frac{\partial^2 \tilde{u}_\epsilon}{\partial x^s \partial x^t}|^2)dx,$$

and

$$\begin{aligned}\int_{B_\delta\setminus B_{Lr_\epsilon}} |x|^2|\frac{\partial^2 \tilde{u}_\epsilon}{\partial x^s \partial x^t}|^2 dx &= \int_{B_\delta\setminus B_{Lr_\epsilon}} |x|^2 \frac{\partial^2 \tilde{u}_\epsilon}{\partial x^t \partial x^t}\frac{\partial^2 \tilde{u}_\epsilon}{\partial x^s \partial x^s}dx + \int_{B_\delta\setminus B_{Lr_\epsilon}} O(|x|\,|\nabla \tilde{u}_\epsilon|\,|\nabla^2 \tilde{u}_\epsilon|)dx \\ &\quad + \int_{\partial(B_\delta\setminus B_{Lr_\epsilon})} |x|^2 \frac{\partial \tilde{u}_\epsilon}{\partial x^t}\frac{\partial^2 \tilde{u}_\epsilon}{\partial x^s \partial x^t}\langle \frac{\partial}{\partial x^t}, \frac{\partial}{\partial r}\rangle ds \\ &\quad + \int_{\partial(B_\delta\setminus B_{Lr_\epsilon})} |x|^2 \frac{\partial \tilde{u}_\epsilon}{\partial x^t}\frac{\partial^2 \tilde{u}_\epsilon}{\partial x^s \partial x^s}\langle \frac{\partial}{\partial x^s}, \frac{\partial}{\partial r}\rangle) ds \\ &= \int_{B_\delta\setminus B_{Lr_\epsilon}} |x|^2 \frac{\partial^2 \tilde{u}_\epsilon}{\partial x^t \partial x^t}\frac{\partial^2 \tilde{u}_\epsilon}{\partial x^s \partial x^s}dx + J(L,\epsilon,\delta).\end{aligned}$$

Hence,

$$2\sum_{k,s,t}\int_{B_\delta\setminus B_{Lr_\epsilon}} |A^{st}\Delta_0 \tilde{u}_\epsilon \frac{\partial^2 \tilde{u}_\epsilon}{\partial x^s \partial x^t}| \leq 4K\int_{B_\delta\setminus B_{Lr_\epsilon}} |x|^2|\Delta_0 \tilde{u}_\epsilon|^2 dx + J(L,\epsilon,\delta).$$

A similar argument as above then gives,

$$\int_{B_\delta\setminus B_{Lr_\epsilon}} \sum_{k,m,s,t} A^{km}A^{st}\frac{\partial^2 \tilde{u}_\epsilon}{\partial x^k \partial x^m}\frac{\partial^2 \tilde{u}_\epsilon}{\partial x^s \partial x^t} \leq K^2\int_{B_\delta\setminus B_{Lr_\epsilon}} |x|^4|\Delta_0 \tilde{u}_\epsilon|^2 dx + J(L,\epsilon,\delta).$$

This proves the Lemma.

$\square$

Lemma 3.4. *There is a function sequence $U_\epsilon \in W^{2,2}(B_\delta \setminus B_{Lr_\epsilon})$ s.t.*

$$U_\epsilon|_{\partial B_\delta} = -2\log\delta + S_0(p) + \bar{u}_\epsilon, \quad U_\epsilon|_{\partial B_{Lr_\epsilon}} = w(L) + m_\epsilon$$

$$\frac{\partial U_\epsilon}{\partial r}|_{\partial B_\delta} = -\frac{2}{\delta}, \quad \frac{\partial U_\epsilon}{\partial r}|_{\partial B_{Lr_\epsilon}} = w'(L)$$

and

$$\int_{B_\delta\setminus B_{Lr_\epsilon}} |\Delta_0(1-B|x|^2)(U_\epsilon - \bar{u}_\epsilon)|^2 dx = \int_{B_\delta\setminus B_{Lr_\epsilon}} |(1-B|x|^2)\Delta_0\tilde{u}_\epsilon|^2 dx + J(L,\epsilon,\delta).$$

Proof. Let u_k' be the solution of

$$\begin{cases} \Delta_0^2 u_\epsilon' = \Delta_0^2 v_\epsilon \\ \frac{\partial u_\epsilon'}{\partial n}|_{\partial B_{2L}} = \frac{\partial v_\epsilon}{\partial n}|_{\partial B_{2L}}, \quad u_\epsilon'|_{\partial B_{2L}} = v_\epsilon|_{\partial B_{2L}} \\ \frac{\partial u_\epsilon'}{\partial n}|_{\partial B_L} = \frac{\partial w}{\partial n}|_{\partial B_L}, \quad u_\epsilon'|_{\partial B_L} = m_\epsilon + w|_{\partial B_L}. \end{cases}$$

We set

$$U_\epsilon' = \begin{cases} u_\epsilon'(\frac{x}{r_\epsilon}) & Lr_\epsilon \le |x| \le 2Lr_\epsilon \\ \tilde{u}_\epsilon(x) & 2Lr_\epsilon \le |x|. \end{cases}$$

It is easy to see that $u_\epsilon' - m_\epsilon$ converges to w smoothly on $B_{2L}\setminus B_L$, we have

$$\lim_{\epsilon\to 0}\int_{B_{2Lr_\epsilon}\setminus B_{Lr_\epsilon}} (1-B|x|^2)^2(|\Delta_0 U_\epsilon'|^2 - |\Delta_0\tilde{u}_\epsilon|^2)dx = 0.$$

Let η be a smooth function which satisfies:

$$\eta(t) = \begin{cases} 1 & t \le 1/2 \\ 0 & t > 2/3 \end{cases}$$

Set $G_\epsilon = \eta(\frac{|x|}{\delta})(\tilde{u}_\epsilon - S_0(p) + 2\log|x|^2 - \bar{u}_\epsilon) - 2\log|x|^2 + S_0(p)$. Recall that $u_\epsilon - \bar{u}_\epsilon$ converges to G_p smoothly on $M \setminus B_{\frac{\delta}{2}}(p)$, we have

$$G_\epsilon \to -2\log|x|^2 + S_0(p) + \eta(\frac{|x|}{\delta})S_1(x), \quad \tilde{u}_\epsilon - G_\epsilon - \bar{u}_\epsilon \to (\eta(\frac{|x|}{\delta}) - 1)S_1(x).$$

Therefore

$$\begin{aligned} &\lim_{\epsilon\to 0}\left|\int_{B_\delta\setminus B_{\delta/2}} |\Delta_0\tilde{u}_\epsilon|^2 dx - \int_{B_\delta\setminus B_{\delta/2}} |\Delta_0 G_\epsilon|^2 dx\right| \\ &\le \sqrt{\textstyle\int_{B_\delta\setminus B_{\delta/2}} |\Delta_0(\eta(\frac{|x|}{\delta}) - 1)S_1(x)|^2 dx \int_{B_\delta\setminus B_{\delta/2}} |\Delta_0(G_p - 2\log|x|^2 + \eta(\frac{|x|}{\delta})S_1(x))|^2 dx} \\ &\le C\sqrt{|\log\delta|}\sqrt{\textstyle\int_{B_\delta\setminus B_{\delta/2}} |\Delta_0\eta(\frac{|x|}{\delta})S_1(x)|^2 dx} \\ &\le C\sqrt{\delta|\log\delta|}. \end{aligned}$$

Now set

$$U_\epsilon = \begin{cases} U'_\epsilon(x) & |x| \leq \frac{\delta}{2} \\ G_\epsilon(x) + \bar{u}_\epsilon & \delta/2 \leq |x| \leq \delta. \end{cases}$$

We then have,

$$\begin{aligned} \int_{B_\delta \setminus B_{L\epsilon}} |(1 - B|x|^2)\Delta_0(U_\epsilon - \bar{u}_\epsilon)|^2 dx &= \int_{B_\delta \setminus B_{Lr_\epsilon}} |\Delta_0(1 - B|x|^2)(U_\epsilon - \bar{u}_\epsilon)|^2 dx \\ &\quad + \int_{B_\delta \setminus B_{Lr_\epsilon}} O(|\nabla U_\epsilon|^2 + |U_\epsilon - \bar{u}_\epsilon|^2) dV_g. \end{aligned}$$

It is easy to check that $\|U_\epsilon - \bar{u}_\epsilon - G_p(E(p,x))\|_{W^{1,2}(B_\delta \setminus B_{Lr_\epsilon})} \to 0$ as $\epsilon \to 0$. Therefore, we proved the lemma.

□

Now, we are going to apply the capacity to derive the lower bound of

$$\int_{B_\delta \setminus B_{Lr_\epsilon}} |\Delta_0(1 - B|x|^2)(U_\epsilon - \bar{u}_\epsilon)|^2 dx.$$

First we need to calculate

$$\inf_{\Phi|_{\partial B_r} = P_1, \Phi|_{\partial B_R} = P_2, \frac{\partial \Phi}{\partial r}|_{\partial B_r} = Q_1, \frac{\partial \Phi}{\partial r}|_{\partial B_R} = Q_2} \int_{B_R \setminus B_r} |\Delta_0 \Phi|^2 dx,$$

where P_1, P_2, Q_1, Q_2 are constants. Obviously, the minimum can be attained by the function Φ which satisfies

$$\begin{cases} \Delta_0^2 \Phi = 0 \\ \Phi|_{\partial B_r} = P_1 \ , \Phi|_{\partial B_R} = P_2 \ , \frac{\partial \Phi}{\partial r}|_{\partial B_r} = Q_1 \ , \frac{\partial \Phi}{\partial r}|_{\partial B_R} = Q_2 \end{cases}$$

Clearly, we can set

$$\Phi = A \log r + Br^2 + \frac{C}{r^2} + D,$$

where A, B, C, D are all constants. Then we have

$$\begin{cases} A \log r + Br^2 + \frac{C}{r^2} + D = P_1 \\ A \log R + BR^2 + \frac{C}{R^2} + D = P_2 \\ \frac{A}{r} + 2Br - 2\frac{C}{r^3} = Q_1 \\ \frac{A}{R} + 2BR - 2\frac{C}{R^3} = Q_2. \end{cases}$$

We have

$$\begin{cases} A = \frac{P_1 - P_2 + \frac{\varrho}{2} r Q_1 + \frac{\varrho}{2} R Q_2}{\log r/R + \varrho} \\ B = \frac{-2P_1 + 2P_2 - rQ_1(1 + \frac{2r^2}{R^2 - r^2} \log r/R) + RQ_2(1 + \frac{2R^2}{R^2 - r^2} \log r/R)}{4(R^2 + r^2)(\log r/R + \varrho)}, \end{cases}$$

where $\varrho = \frac{R^2 - r^2}{R^2 + r^2}$. Furthermore,

$$\int_{B_R \setminus B_r} |\Delta_0 \Phi|^2 dx = -8\pi^2 A^2 \log r/R + 32\pi^2 AB(R^2 - r^2) + 32\pi^2 B^2 (R^4 - r^4)$$

In our case, $R=\delta$, $r=Lr_\epsilon$, $P_1=m_\epsilon-\bar{u}_\epsilon+w(L)+O(r_\epsilon\bar{u}_\epsilon)$, $P_2=-2\log\delta+S_0(p)+O(\delta\log\delta)$, $Q_1=\frac{2\lambda L}{r_\epsilon(1+\lambda L^2)}$, $Q_2=-\frac{2}{\delta}+O(\delta\log\delta)$. If we define

$$\begin{aligned}
N(L,\epsilon,\delta) &= w(L)+2\log\delta-S_0-\tfrac{\varrho}{2}\tfrac{2\lambda L^2}{1+\lambda L^2}\\
&= w(L)+2\log\delta-S_0-2+O(\delta\log\delta)+O(\tfrac{1}{L^2})+O(Lr_\epsilon),
\end{aligned}$$

and

$$P=\log\delta-\log L,$$

then we have

$$\begin{aligned}
A^2(-\log Lr_\epsilon/\delta) &= (\tfrac{m_\epsilon-\bar{u}_\epsilon+N(L,\epsilon,\delta)}{m_\epsilon+P-\varrho})^2(m_\epsilon+P)\\
&= (1+\tfrac{P-\varrho}{m_\epsilon})^{-2}(1+\tfrac{P}{m_\epsilon})m_\epsilon(1-\tfrac{\bar{u}_\epsilon}{m_\epsilon}+\tfrac{N(L,\epsilon,\delta)}{m_\epsilon})^2\\
&= (1-2\tfrac{P-\varrho}{m_\epsilon}+O(\tfrac{1}{m_\epsilon^2}))(1+\tfrac{P}{m_\epsilon})m_\epsilon\\
&\quad \Big[(1-\tfrac{\bar{u}_\epsilon}{m_\epsilon})^2+2(1-\tfrac{\bar{u}_\epsilon}{m_\epsilon})\tfrac{N(L,\epsilon,\delta)}{m_\epsilon}+O(\tfrac{1}{m_\epsilon^2})+O(e^{-m_\epsilon}m_\epsilon)\tfrac{\bar{u}_\epsilon}{m_\epsilon}\Big]\\
&= m_\epsilon(1-\tfrac{\bar{u}_\epsilon}{u_\epsilon})^2+2(1-\tfrac{\bar{u}_\epsilon}{m_\epsilon})N(L,\epsilon,\delta)-(P-2\varrho)(1-\tfrac{\bar{u}_\epsilon}{m_\epsilon})^2\\
&\quad +O(\tfrac{1}{m_\epsilon})(1-\tfrac{\bar{u}_\epsilon}{m_\epsilon})^2+O(\tfrac{1}{m_\epsilon}),
\end{aligned}$$

and

$$A=-\frac{m_\epsilon-\bar{u}_\epsilon+N(L,\epsilon,\delta)}{m_\epsilon-\log L+\log\delta+\varrho}=-(1-O(\frac{1}{m_\epsilon}))^{-1}(1-\frac{\bar{u}_\epsilon}{m_\epsilon}+O(\frac{1}{m_\epsilon}))=-1+\frac{\bar{u}_\epsilon}{m_\epsilon}+O(\frac{1}{m_\epsilon}).$$

Notice that $r_\epsilon m_\epsilon\to 0$ as $\epsilon\to 0$, we have

$$\begin{aligned}
B &= \tfrac{-2m_\epsilon+2\bar{u}_\epsilon+O(1)+(2\frac{2\delta^2}{\delta^2-(Lr_\epsilon)^2}+O(\delta\log\delta))m_\epsilon}{4(\delta^2+(Lr_\epsilon)^2)(\log L-m_\epsilon-\log\delta+\varrho)}\\
&= -\tfrac{1}{2\delta^2}(1+\tfrac{\bar{u}_\epsilon}{m_\epsilon}+O(\tfrac{1}{m_\epsilon}))(1-O(\tfrac{1}{m_\epsilon}))^{-1}\\
&= -\tfrac{1}{2\delta^2}(1+\tfrac{\bar{u}_\epsilon}{m_\epsilon}+O(\tfrac{1}{m_\epsilon})).
\end{aligned}$$

It concludes that

$$\begin{aligned}
\int_{B_\delta\setminus B_{Lr_\epsilon}}|\Delta_0(1-B|x|^2)(U_\epsilon-\bar{u}_\epsilon)|^2dx &\geq 8\pi^2 m_\epsilon(1-\tfrac{\bar{u}_\epsilon}{m_\epsilon})^2+16\pi^2(1-\tfrac{\bar{u}_\epsilon}{m_\epsilon})N(L,\epsilon,\delta)\\
&\quad -8\pi^2(P-2\varrho)(1-\tfrac{\bar{u}_\epsilon}{m_\epsilon})^2\\
&\quad +16\pi^2(1-\tfrac{\bar{u}_\epsilon}{m_\epsilon})(1+\tfrac{\bar{u}_\epsilon}{m_\epsilon})+8\pi^2(1+\tfrac{\bar{u}_\epsilon}{m_\epsilon})^2\\
&\quad +O(\tfrac{1}{m_\epsilon})(1-\tfrac{\bar{u}_\epsilon}{m_\epsilon})^2+O(\tfrac{1}{m_\epsilon})+J_6(L,\epsilon,\delta).
\end{aligned}$$

Using the fact that $\bar{u}_\epsilon\leq C$, we have

$$(8\pi^2-\epsilon)\bar{u}_\epsilon>8\pi^2\bar{u}_\epsilon+\epsilon C.$$

Therefore

$$\begin{aligned}
II_\epsilon(u_\epsilon) &\geq \textstyle\int_{B_{Lr_\epsilon}(x_\epsilon)}|\Delta_g u_\epsilon|^2dV_g+\int_{B_\delta\setminus B_{Lr_\epsilon}}|\Delta_0(1-|B|^2)(U_\epsilon-\bar{u}_\epsilon)|^2dx+8\pi^2\bar{u}_\epsilon\\
&\quad \textstyle +\int_{M\setminus B_\delta(x_0)}\langle G_p,G_p\rangle+4\int_M\tilde{Q}G_p dV_g+J(L,\epsilon,\delta)\\
&\geq 8\pi^2(m_\epsilon+C_1)(1+\tfrac{\bar{u}_\epsilon}{m_\epsilon})^2+C_2(1+\tfrac{\bar{u}_\epsilon}{m_\epsilon})+C_3.
\end{aligned}$$

where C_1, C_2, C_3 are some constants. Note that since $II_\epsilon(u_\epsilon) < \infty$, we must have $(1+\frac{\bar{u}_\epsilon}{m_\epsilon}) \to 0$ as $\epsilon \to 0$, i.e. $\frac{\bar{u}_\epsilon}{m_\epsilon} \to -1$.

Consequently we have

$$\begin{aligned}
&\textstyle\int_{B_\delta \setminus B_{Lr_\epsilon}} |\Delta_0(1-B|x|^2)(U_\epsilon - \bar{u}_\epsilon)|^2 dx + 8\pi^2 \bar{u}_\epsilon \\
&\quad \geq 8\pi^2 m_\epsilon (1+\tfrac{\bar{u}_\epsilon}{m_\epsilon})^2 + 16\pi^2 N(L,\epsilon,\delta)(1-\tfrac{\bar{u}_\epsilon}{m_\epsilon}) - 8\pi^2(\log\delta - \log L - 2\varrho)(1-\tfrac{\bar{u}_\epsilon}{m_\epsilon})^2 \\
&\qquad +J(L,\epsilon,\delta) \\
&\quad \geq 16\pi^2(1-\tfrac{\bar{u}_\epsilon}{m_\epsilon})N(L,\epsilon,\delta) - 8\pi^2(\log\delta - \log L - 2\varrho)(1-\tfrac{\bar{u}_\epsilon}{m_\epsilon})^2 + J(L,\epsilon,\delta).
\end{aligned} \tag{3.2}$$

Since we have

$$\Delta_0 w = \frac{4\lambda^2|x|^2}{(1+\lambda|x|^2)^2} - \frac{8\lambda}{1+\lambda|x|^2},$$

a direct calculation yields that

$$\int_{B_L} |\Delta_0 w|^2 dx = 16\pi^2 \log(1+\lambda L^2) + \frac{8\pi^2}{3} + O(\frac{\log L}{L^2}).$$

On the other hand, it is obvious to see that,

$$\int_{B_\delta(x_\epsilon)} |\nabla u_\epsilon|^2 \to \int_{B_\delta(x_\epsilon)} |\nabla G_p|^2 = O(\delta \log\delta), \tag{3.3}$$

and

$$\begin{aligned}
\int_{M\setminus B_\delta(x_0)} \langle G_p, G_p\rangle dV_g &= \int_{M\setminus B_\delta(x_0)} G_p P_g G_p dV_g - \int_{\partial B_\delta} \frac{\partial G_p}{\partial r}\Delta_g G_p dV_g + \int_{\partial B_\delta} G_p \frac{\partial \Delta G_p}{\partial r} dV_g \\
&\quad + \int_{\partial B_\delta} (\frac{2}{3} RG\frac{\partial G}{\partial r} - 2G Ric(dG, dr)) dS_g \\
&= -2\int_M Q_g G_p dV_g - 16\pi^2 + 16\pi^2(-2\log\delta + S_0(p)) + O(\delta\log\delta).
\end{aligned} \tag{3.4}$$

Together with Lemma 3.3, Lemma 3.4, (3.2), (3) and (3.4), we have

$$\begin{aligned}
\lim_{\epsilon\to 0} II_\epsilon &\geq 32\pi^2 \lim_{\epsilon\to 0} N(L,\epsilon,\delta) - 32\pi^2(\log\delta - \log L - 2) + 16\pi^2\log(1+\lambda L^2) \\
&\quad + \tfrac{8\pi^2}{3} + (-2\log\delta + S_0(p))16\pi^2 + 2\int_M Q_g G_p dV_g - 8\pi^2 \log 8\pi^2 + O(\delta\log\delta) + O(\frac{\log L}{L^2}) \\
&= -16\pi^2 \log\tfrac{1+\lambda L^2}{L^2} + \tfrac{8\pi^2}{3} - 16\pi^2 S_0(p) - 16\pi^2 + 2\int_M Q_g G_p dV_g - 8\pi^2\log 8\pi^2 \\
&\quad + O(\delta\log\delta) + O(\tfrac{\log L}{L^2}).
\end{aligned}$$

Letting first $\delta \to 0$, then $L \to +\infty$, we get

$$\lim_{\epsilon\to 0} II_\epsilon \geq -16\pi^2\log\lambda - 8\pi^2\log 8\pi^2 - 16\pi^2 S_0 + (8/3-16)\pi^2 + 2\int_M Q_g G_p dV_g.$$

This shows the first part of Theorem 1.1, that is

$$\inf_{u\in W^{2,2}(M)} II(u) \geq \inf_{p\in M} \Lambda_g(\tilde{Q}, p).$$

The second part

$$\inf_{u\in W^{2,2}(M)} II(u) \le \inf_{p\in M} \Lambda_g(\tilde{Q},p)$$

follows from the proof of Theorem 1.2 in next section.

To end this section, we will prove a conformal property of $\Lambda_g(\tilde{Q},p)$.

Lemma 3.5. *Let $\tilde{g}\in[g]$: $\tilde{g}=e^{2v}g$ for some $v\in C^\infty(M)$, we have*

$$II_{\tilde{g}}(u) = II_g(u+v) - \int_M \langle v,v\rangle dV_g.$$

If we set

$$P_{\tilde{g}}\tilde{G}_y + 2Q_{\tilde{g}} = 16\pi^2\delta_y,$$

then $\tilde{G}_y = G_y - v$. Moreover, for any y, we have

$$2\int_M Q_{\tilde{g}}\tilde{G}_y dV_{\tilde{g}} - 16\pi^2\tilde{S}_0(y) = 2\int_M Q_g G_y dV_g - 16\pi^2 S_0(y) - \int_M \langle v,v\rangle dV_g.$$

Proof. Since $P_{\tilde{g}} = e^{-4v}P_g$, $2Q_{\tilde{g}} = e^{-4v}(P_g v + 2Q_g)$, we get

$$\begin{aligned}
II_{\tilde{g}}(u) &= \int_M \langle u,u\rangle dV_g + 2\int_M (P_g v + 2Q_g)u dV_g - 8\pi^2\log\int_M \tilde{Q}e^{4(u+v)}dV_g \\
&= \int_M \langle u+v,u+v\rangle dV_g + 4\int_M Q_g u dV_g - 8\pi^2\log\int_M \tilde{Q}e^{4(u+v)}dV_g - \int_M \langle v,v\rangle dV_g \\
&= II_g(u+v) - \int_M \langle v,v\rangle dV_g.
\end{aligned}$$

On the other hand, we have

$$P_{\tilde{g}}(G-v) + 2Q_{\tilde{g}} = e^{-4v}(P_g G + 2Q_g) = 16\pi^2 e^{-4v}\delta_{y,g} = 16\pi^2\delta_{y,\tilde{g}}.$$

Since $dist_{\tilde{g}}(y,x) = e^{v(y)}dist_g(y,x) + O(dist_g(y,x))^2$, we have

$$\begin{aligned}
\tilde{G}_y &= G_y - v \\
&= -2\log dist_g(y,x) + S_0(y) - v(y) + O(dist(y,x)) \\
&= -2\log dist_{\tilde{g}}(y,x) + v(y) + S_0(y) + O(dist(y,x)).
\end{aligned}$$

Thus $\tilde{S}_0(y) = S_0(y) + v(y)$. Moreover, we have

$$\begin{aligned}
\int_M Q_{\tilde{g}}\tilde{G}_y dV_{\tilde{g}} &= \int_M (P_g v + 2Q_g)(G_y - v)dV_g \\
&= (\int_M G_y P_g v dV_g + 2\int_M Q_g v dV_g) + 2\int_M Q_g G_y dV_g - \int_M vP_g v dV_g \\
&= 16\pi^2 v(y) + 2\int_M Q_g G_y dV_g - \int_M vP_g v dV_g,
\end{aligned}$$

this proves the lemma.

□

4 Testing function

In this section we will construct a blow up sequence ϕ_ϵ s.t.

$$II(\phi_\epsilon) < \inf_{x\in M} \Lambda(x).$$

We use standard notation from [L-P]. In a local coordinate system $\{x^i\}$, we denote

$$R_{ijkl} =< R(\partial_k, \partial_l)\partial_j, \partial_i >, \quad R_{ij} = -g^{jk}R_{ijkl},$$

where R is the curvature operator, defined as follows,

$$R(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$$

Suppose that p' is a point such that $\Lambda(p') = \inf_{x\in M}\Lambda(x)$.

We know that, locally we have

$$g_{pq} = \delta_{pq}+\frac{1}{3}R_{pijq}(p')x^ix^j+\frac{1}{6}R_{pijq,k}(p')x^ix^jx^k+(\frac{1}{20}R_{pijq,kl}+\frac{2}{45}R_{pijm}(p')R_{qklm}(p'))x^ix^jx^kx^l+O(r^5).$$

$$|g| = 1 - \frac{1}{3}R_{ij}x^{ij} - \frac{1}{6}R_{ij,k}(p')x^{ijk} - (\frac{1}{20}R_{ij,kl}(p') + \frac{1}{90}R_{hijm}(p')R_{hklm}(p'))x^ix^jx^kx^m + O(r^5)$$

In the sequel, let us denote

$$x^{i_1\cdots i_m}_{j_1\cdots j_n} = x^{i_1\cdots i_m j_1\cdots j_n}, \quad and \quad \alpha^{i_1\cdots i_m}_{j_1\cdots j_n} = \frac{1}{2\pi^2}\int_{S^3} x^{i_1\cdots i_m j_1\cdots j_n}ds,$$

then around the point p' we write

$$g^{km} = \delta^{km} + M^{km} = \delta^{km} + M^{ij}_{km}x^{km} + M^{ij}_{kms}x^{kms} + M^{ij}_{kmst}x^{kmst} + O(r^5)$$

$$M = M^{ij}\delta_{ij} = M_{km}x^{km} + M_{kms}x^{kms} + M_{kmst}x^{kmst} + O(r^5),$$

$$\sqrt{|g|} = 1 - \frac{1}{6}R_{ij}x^{ij} + K_{ijk}x^{ijk} + K_{ijkm}x^{ijkm} + O(r^5).$$

$$N^k = -g^{ij}\Gamma^k_{ij} = N^k_i x^i + N^k_{ij}x^{ij} + N^k_{ijm}x^{ijm} + O(r^5).$$

It is easy to check that $M^{ij}_{km} = -\frac{1}{3}R_{ikmj}(p')$, $M_{km} = \frac{1}{3}R_{ij}(p')$ and $N^k_i = -\frac{2}{3}R_{ik}(p')$.

We prove the following lemma.

Lemma 4.1. *We have*

$$\frac{1}{18}R_{ij}(p')R_{km}(p')\alpha^{ijkm} + N^m_{ijk}\alpha^{ijk}_m + M_{ijkm}\alpha^{ijkm} = 4K_{ijkm}\alpha^{ijkm}. \tag{4.1}$$

Proof. We have, for any small $t > 0$,

$$\begin{aligned}\int_{B_t}\Delta_g r^2 dV_g &= \int_{B_t}(8 - \frac{2}{3}R_{ij}x^{ij} + 2M_{ijk}x^{ijk} + 2M_{ijkm}x^{ijkm} + 2N^k_{ij}x^{ij}_k + 2N^p_{ijk}x^{ijk}_p) \\ &\quad \times(1 - \tfrac{1}{6}R_{ij}x^{ij} + K_{ijk}x^{ijk} + K_{ijkm}x^{ijkm})dx + o(t^8) \\ &= 4\pi^2t^4 - 2R_{ij}\alpha^{ij}\times 2\pi^2\tfrac{t^6}{6} \\ &\quad +(\tfrac{1}{9}R_{ij}R_{km}\alpha^{ijkm} + 2M_{ijkm}\alpha^{ijkm} + 2N^p_{ijk}\alpha^{ijk}_p + 8K_{ijkm}\alpha^{ijkm})2\pi^2\tfrac{t^8}{8} + o(t^8),\end{aligned}$$

on the other hand, we have

$$\begin{aligned}\int_{\partial B_t} 2r ds_g &= \int_{\partial B_t} 2r(1-\frac{1}{6}R_{ij}x^{ij}+K_{ijkm}x^{ijkm}+O(r^5))ds_0\\ &= 4\pi^2t^4-4\pi^2\tfrac{R_{ij}}{6}\alpha^{ij}t^6+2K_{ijkm}\alpha^{ijkm}2\pi^2t^8+o(t^8).\end{aligned}$$

Now the conclusion follows from the Stokes' theorem.

□

Note that locally, we may write (see Lemma 6.1 in the appendix),

$$G_{p'} = -2\log r + S,$$

with

$$S = S_0(p') + a_ix^i + \frac{a_{ij}}{2}x^{ij} + O(r^{2+\alpha}).$$

We define

$$\varphi_\epsilon = -\log(1+\lambda|\frac{x}{\epsilon}|^2) + C_\epsilon + \mu|x|^2, \quad x \in B_{L\epsilon}$$

where

$$\mu = -\frac{1}{L^2\epsilon^2(1+\lambda L^2)}, \quad \lambda = \frac{\sqrt{3\tilde{Q}(p')}}{12}$$

and

$$C_\epsilon = \log(1+\lambda L^2) - 2\log L\epsilon - \mu L^2\epsilon^2.$$

We set

$$\phi_\epsilon = \begin{cases} G+\varphi_\epsilon+2\log r & x\in B_{L\epsilon}\\ G & x\notin B_{L\epsilon},\end{cases}$$

then, in $B_{L\epsilon}$, we have

$$\phi_\epsilon = -\log(1+\lambda|\frac{x}{\epsilon}|^2) + C_\epsilon + S + \mu|x|^2.$$

Hence, it is easy to check that $\phi_\epsilon \in W^{2,p}(M)$ for any $p>0$.

We write

$$\begin{aligned}II(\phi_\epsilon) :&= \int_M \langle\phi_\epsilon,\phi_\epsilon\rangle dV_g + 4\int_M Q_g\phi_\epsilon dV_g - 8\pi^2\log\int_M \tilde{Q}e^{4\phi_\epsilon}dV_g\\ &= II_1 + II_2 + II_3\end{aligned}$$

First we will calculate the term II_3. In the small neighborhood around the point p', we set

$$\tilde{Q} = \tilde{Q}(p') + b_ix^i + \frac{b_{ij}}{2}x^{ij} + O(r^3),$$

then we have

$$\begin{aligned}\tilde{Q}e^{4\phi_\epsilon}\sqrt{|g|} &= \tfrac{e^{4C_\epsilon+4S_0}}{\epsilon^4(1+\lambda|\frac{x}{\epsilon}|^2)^4}[(1+4a_ix^i+2a_{ij}x^{ij}+8a_ia_jx^{ij}+4\mu r^2)\tilde{Q}(p')+b_ix^i+\tfrac{b_{ij}}{2}x^{ij}+4a_ib_ix^{ij}\\ &\quad +O(r^{2+\alpha})+O(\tfrac{r^2\epsilon^2}{L^8})](1-\tfrac{R_{ij}x^{ij}}{6}+O(r^3))\\ &= \tfrac{e^{4C_\epsilon+4S_0}}{\epsilon^4(1+\lambda|\frac{x}{\epsilon}|^2)^4}[(1+4a_ix^i+2a_{ij}x^{ij}+8a_ia_jx^{ij}+4\mu r^2-\tfrac{R_{ij}x^{ij}}{6})\tilde{Q}(p')+b_ix^i+\tfrac{b_{ij}}{2}x^{ij}+4a_ib_ix^{ij}\\ &\quad +O(r^{2+\alpha})+O(\tfrac{r^2}{L^8})].\end{aligned}$$

Therefore, by using the symmetry of the ball and the fact that $\alpha_{ij} = \frac{1}{4}\delta_{ij}$, we have

$$\begin{aligned}\int_{B_{L\epsilon}} \tilde{Q} e^{4\phi_\epsilon}\sqrt{|g|}dV_g &= 2\pi^2 e^{4C_\epsilon+4S_0(p')}\epsilon^4\int_0^L \frac{1}{(1+\lambda r^2)^4}[\tilde{Q}(p')(1+\epsilon^2 r^2(\sum_i(\frac{a_{ii}}{2}+2a_i^2)+4\mu-\frac{R(p')}{24}) \\ &\quad +\sum_i(a_ib_i+\tfrac{b_{ii}}{8})\epsilon^2 r^2 + O(\epsilon r)^{2+\alpha}+O(\tfrac{r^2}{L^4})]r^3dr.\end{aligned}$$

A direct calculation then yields that

$$2\pi^2\int_0^L \frac{r^3dr}{(1+\lambda r^2)^4} = \frac{\pi^2}{6\lambda^2}+O(\frac{1}{L^4}),$$

$$2\pi^2\int_0^L \frac{r^5dr}{(1+\lambda r^2)^4} = \frac{\pi^2}{3\lambda^3}+O(\frac{1}{L^2}),$$

and

$$4\mu\epsilon^2\times 2\pi^2\int_0^L \frac{r^5dr}{(1+\lambda r^2)^4} = O(\frac{1}{L^4}).$$

Hence we get

$$\begin{aligned}\int_{B_{L\epsilon}} \tilde{Q} e^{4\phi_\epsilon}\sqrt{|g|}dx &= e^{4C_\epsilon+4S_0}\epsilon^4[8\pi^2-\tfrac{24\pi^2}{\lambda^2L^4}+\tfrac{\pi^2}{3\lambda^3}\epsilon^2(\sum_i(\tfrac{a_{ii}}{2}+2a_i^2)\tilde{Q}(p')-\tfrac{R(p')}{24}\tilde{Q}(p') \\ &\quad +\sum_i(a_ib_i+\tfrac{b_{ii}}{8}))+O(\tfrac{1}{L^4})+O(\epsilon^{2+\alpha})+O(\tfrac{\epsilon^2}{L^2})].\end{aligned}$$

On the other hand, it is not difficult to check that

$$\begin{aligned}\int_{M\setminus B_{L\epsilon}} \tilde{Q} e^{4\phi_\epsilon}\sqrt{|g|}dx &= \int_{L\epsilon}^{\delta}\tilde{Q}(p')\frac{e^{4S_0}}{r^5}2\pi^2dr+O(\frac{1}{L^2\epsilon^2}) \\ &= e^{4C_\epsilon+4S_0}\epsilon^4(\tfrac{24\pi^2}{\lambda^2L^4}+O(\tfrac{\epsilon^2}{L^2})).\end{aligned}$$

In sum, we have

$$\begin{aligned}8\pi^2\log\int_M \tilde{Q} e^{4\phi_\epsilon}\sqrt{|g|}dx &= 8\pi^2[\log 8\pi^2+4(C_\epsilon+\log\epsilon+S_0)] \\ &\quad +\tfrac{\pi^2}{3\lambda^3}[\tilde{Q}(p')\sum_i(\tfrac{a_{ii}}{2}+2a_i^2)+\sum_i(a_ib_i+\tfrac{b_{ii}}{8})-\tfrac{R(p')}{24}\tilde{Q}(p')]\epsilon^2 \\ &\quad +O(\epsilon^{2+\alpha})+O(\tfrac{\epsilon^2}{L^2})+O(\tfrac{1}{L^4}).\end{aligned} \tag{4.2}$$

The next, we calculate II_1: First of all, we have

$$\begin{aligned}\int_M \langle\phi_\epsilon,\phi_\epsilon\rangle dV_g &= \int_M\langle G,\phi_\epsilon\rangle dV_g+\int_{B_{L\epsilon}}\langle\varphi_\epsilon+2\log r,\phi_\epsilon\rangle dV_g \\ &= 16\pi^2(C_\epsilon+S_0(p'))-2\int_M Q\phi_\epsilon dV_g+\int_{B_{L\epsilon}}\langle\varphi_\epsilon+2\log r,\varphi_\epsilon+S\rangle dV_g.\end{aligned} \tag{4.3}$$

We set η to be a cut-off function which is 0 at 1 and 1 in $[0,1/4]$ with $\eta'(1)=1$, and

$$h_\tau = \begin{cases}\eta(\frac{|x|}{\tau})+\log\tau & |x|\le\tau \\ \log r & |x|\ge\tau.\end{cases}$$

Then for fixed ϵ and L, we have

$$\lim_{\tau\to 0}\int_{B_{L\epsilon}}\langle\varphi_\epsilon+2h_\tau,\varphi_\epsilon+S\rangle dV_g=\int_{B_{L\epsilon}}\langle\varphi_\epsilon+2\log r,\varphi_\epsilon+S\rangle dV_g.$$

On the other hand, we have

$$\begin{aligned}\int_{B_{L\epsilon}}\langle\varphi_\epsilon+2h_\tau,\varphi_\epsilon+S\rangle dV_g &= \int_{B_{L\epsilon}}\langle\varphi_\epsilon+2h_\tau,G\rangle dV_g+\int_{B_{L\epsilon}}\langle\varphi_\epsilon+2h_\tau,\varphi_\epsilon+2\log r\rangle dV_g\\ &= 16\pi^2C_\epsilon+32\pi^2\eta(0)+32\pi^2\log\tau-2\int_{B_{L\epsilon}}Q_g(\varphi_\epsilon+2h_\tau)\\ &\quad+\int_{B_{L\epsilon}}\langle\varphi_\epsilon,\varphi_\epsilon\rangle dV_g+\int_{B_{L\epsilon}}\langle\varphi_\epsilon,2\log r+2h_\tau\rangle dV_g\\ &\quad+\int_{B_{L\epsilon}}\langle 2\log r,2h_\tau\rangle dV_g.\end{aligned}$$

Therefore we get

$$\begin{aligned}&\int_{B_{L\epsilon}}\langle\varphi_\epsilon+2\log r,\varphi_\epsilon+S\rangle dV_g\\ &= 32\pi^2\eta(0)-2\int_{B_{L\epsilon}}Q_g(\varphi_\epsilon+2\log r)+\int_{B_{L\epsilon}}\langle\varphi_\epsilon,\varphi_\epsilon\rangle dV_g\\ &\quad+\int_{B_{L\epsilon}}\langle\varphi_\epsilon,4\log r\rangle dV_g+\lim_{\tau\to 0}(\int_{B_{L\epsilon}}\langle 2\log r,2h_\tau\rangle dV_g+32\pi^2\log\tau)\\ &= 32\pi^2\eta(0)-2\int_{B_{L\epsilon}}Q_g(\varphi_\epsilon+2\log r)+\int_{B_{L\epsilon}}\Delta_g\varphi_\epsilon\Delta_g\varphi_\epsilon dV_g\\ &\quad+4\int_{B_{L\epsilon}}\Delta_g\varphi_\epsilon\Delta_g\log r dV_g+\lim_{\tau\to 0}(\int_{B_{L\epsilon}}\Delta_g 2\log r\Delta_g 2h_\tau dV_g+32\pi^2\log\delta)\\ &\quad+\int_{B_{L\epsilon}}\frac{2}{3}R\langle d(\varphi_\epsilon+2\log r),d(\varphi_\epsilon+2\log r)\rangle dV_g\\ &\quad-\int_{B_{L\epsilon}}2Ric(d(\varphi_\epsilon+2\log r),d(\varphi_\epsilon+2\log r))dV_g.\end{aligned}\tag{4.4}$$

By a simple calculation, one gets

$$\begin{aligned}\int_{B_\tau}(\Delta_g 2\log r)\Delta_g(2h_\tau)dV_g &= \int_{B_\tau}\Delta_0(2\log r)\Delta_0(2\eta(\frac{|x|}{\tau}))dx+O(\tau)\\ &= -32\pi^2\eta(0)+16\pi^2+O(\tau).\end{aligned}\tag{4.5}$$

To compute $\int_{B_{L\epsilon}\setminus B_\delta}\Delta_g\log r\Delta_g\log r$, we first verify that, for any smooth function f, g which are smooth in (t_0,t_1), we have

$$\begin{aligned}\Delta_g f(r) &= (\delta_{km}+M_{ij}^{km}x^{ij}+M_{ijs}^{km}x^{ijs}+M_{ijst}^{km}x^{ijst}+O(r^5))(f''\tfrac{x_{km}}{r^2}+f'\tfrac{\delta_{km}}{r}-f'\tfrac{x_{km}}{r^3})+N^k\tfrac{x_k}{r}f'\\ &= f''+f'(\tfrac{3}{r}-\tfrac{R_{ij}x^{ij}}{3r}+\tfrac{M_{ijk}x^{ijk}+N_{ij}^k x_k^{ij}}{r}+\tfrac{M_{ijkm}x^{ijkm}+N_{ijk}^m x_m^{ijk}}{r})+O(r^5|f''|)+O(r^4|f'|).\end{aligned}$$

Here, we use the fact that $M_{ij}^{km}x_{km}^{ij} = M_{ijst}^{km}x_{km}^{ijst} = 0$. Then, applying Lemma 4.1, we get

$$\begin{aligned}
&\int_{B_{t_1}\setminus B_{t_0}} \Delta_g f(|x|)\Delta_g g(|x|)dV_g \\
&= \int_{t_0}^{t_1} f''g''(1-\frac{R}{24}r^2 + K_{ijkm}\alpha^{ijkm}r^4)2\pi^2r^3dr \\
&\quad + \int_{t_0}^{t_1}(f'g''+f''g')\frac{1}{r}(3-\frac{5R}{24}r^2+7K_{ijkm}\alpha^{ijkm}r^4)2\pi^2r^3dr \\
&\quad + \int_{t_0}^{t_1} f'g'\frac{1}{r^2}(9+33K_{ijkm}\alpha^{ijkm}r^4-\frac{7R}{8}r^2+\frac{1}{9}R_{ij}R_{km}\alpha^{ijkm}r^2)2\pi^2r^3dr \\
&\quad + \int_{t_0}^{t_1}\big(O(r^8|f''g''|)+O(r^7(|f''g'|+|f'||g''|))+O(r^6|f'g'|)\big) \\
&= \int_{t_0}^{t_1}(f''g''+(f'g''+f''g')\frac{3}{r}+f'g'\frac{9}{r^2})2\pi^2r^3 \\
&\quad + R\int_{t_0}^{t_1}(-f''g''\frac{r^2}{24}-\frac{5r}{24}(f'g''+f''g')-\frac{7}{8}f'g')2\pi^2r^3 \\
&\quad + K_{ijkm}\alpha^{ijkm}\int_{t_0}^{t_1}(f''g''r^4+7(f'g''+f''g')r^3+33f'g'r^2)2\pi^2r^3dr \\
&\quad + R_{ij}R_{km}\alpha^{ijkm}\int_{t_0}^{t_1}\frac{1}{9}f'g'r^22\pi^2r^3dr \\
&\quad + \int_{t_0}^{t_1}\big(O(r^8|f''g''|)+O(r^7(|f''g'|+|f'||g''|))+O(r^6|f'g'|)\big)\,dr.
\end{aligned} \tag{4.6}$$

Then, choosing $f = g = 2\log r$, $t_1 = L\epsilon$, $t_0 = \tau$, we get

$$\begin{aligned}
\int_{B_{L\epsilon}\setminus B_\tau}\Delta_g(2\log r)\Delta_g(2h_\tau)dV_g &= \int_{B_{L\epsilon}\setminus B_\tau}\Delta_g(2\log r)\Delta_g(2\log r)dV_g \\
&= 40K_{ijkm}\alpha^{ijkm}\pi^2(L\epsilon)^4 + \tfrac{2\pi^2}{9}R_{ij}R_{km}\alpha^{ijkm}(L\epsilon)^4 \\
&\quad -2R\pi^2(L\epsilon)^2 + 32\pi^2\log L\epsilon - 32\pi^2\log\tau \\
&\quad +O(\tau) + O(L\epsilon)^5.
\end{aligned} \tag{4.7}$$

Now we will calculate the term $\int_{B_{L\epsilon}}\Delta_g\varphi_\epsilon\Delta_g(\varphi_\epsilon + 4\log r)dV_g$: In (4.6), we choose $f = \varphi_\epsilon$, $g = \varphi_\epsilon + 4\log r$, $t_0 = 0$, $t_1 = L\epsilon$ then we get

$$\begin{aligned}
\int_{B_{L\epsilon}}\Delta_g\varphi_\epsilon\Delta_g(\varphi_\epsilon+4\log r)dV_g &= -\tfrac{88}{3}\pi^2 + \tfrac{16\pi^2}{\lambda L^2} - 16\pi^2\log(1+\lambda L^2) \\
&\quad -R\epsilon^2\tfrac{8\pi^2}{9\lambda} + 2\pi^2R(L\epsilon)^2 \\
&\quad -40K_{ijkm}\alpha^{ijkm}\pi^2(L\epsilon)^4 - \tfrac{2\pi^2}{9}R_{ij}R_{km}\alpha^{ijkm}(L\epsilon)^4 \\
&\quad +O(\epsilon^4L^2) + \tfrac{\epsilon^2}{L^2} + O(L\epsilon)^5.
\end{aligned} \tag{4.8}$$

By a direct calculation, we have

$$\begin{aligned}
&\int_{B_{L\epsilon}} \frac{2}{3} R(\nabla_g(\varphi_\epsilon + 2\log r), \nabla_g(\varphi_\epsilon + 2\log r)) dV_g \\
&\quad = \tfrac{2}{3}\int_0^{L\epsilon} R(p')(\frac{2\epsilon^2}{(\epsilon^2+\lambda r^2)r} + 2\mu r)^2 2\pi^2 r^3 \\
&\quad + \tfrac{2}{3}\int_{B_{L\epsilon}} (R_{,i}(p')x^i + O(r^2))(\frac{2\epsilon^2}{(\epsilon^2+\lambda r^2)r} + 2\mu r)^2(1+O(r^3))dx \\
&\quad = \tfrac{8}{3\lambda} R(p')\pi^2\epsilon^2 + \int_{B_{L\epsilon}} (\frac{2\epsilon^2}{(\epsilon^2+\lambda r^2)r} + 2\mu r)^2 O(r^2) dx \\
&\quad = \tfrac{8}{3\lambda} R(p')\pi^2\epsilon^2 + O(\epsilon^4 L^2) + O(\tfrac{\epsilon^2}{L^2}),
\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}
&\int_{B_{L\epsilon}} 2Ric(\nabla_g(\varphi_\epsilon + 2\log r), \nabla_g(\varphi_\epsilon + 2\log r))] dV_g \\
&\quad = \tfrac{1}{2} R(p') \int_0^{L\epsilon} (\frac{2\epsilon^2}{(\epsilon^2+\lambda r^2)r} + 2\mu r)^2 2\pi^2 r^3 dr \\
&\quad + 2\int_{B_{L\epsilon}} g^{is}g^{jt}(R_{ij,k}(p')x^k + O(r^2))(\frac{2\epsilon^2}{(\epsilon^2+\lambda r^2)r^2} + 2\mu)^2 x_{st}(1+O(r^3))dx \\
&\quad = \tfrac{2}{\lambda} R(p')\pi^2\epsilon^2 + 2\int_{B_{L\epsilon}} (R_{ij,k}(p')x^k + O(r^2))(\frac{2\epsilon^2}{(\epsilon^2+\lambda r^2)r^2} + 2\mu)^2 x^{ij}(1+O(r^3))dx \\
&\quad = \tfrac{2}{\lambda} R(p')\pi^2\epsilon^2 + \int_{B_{L\epsilon}} (\frac{2\epsilon^2}{(\epsilon^2+\lambda r^2)r^2} + 2\mu)^2 O(r^4) dx \\
&\quad = \tfrac{2}{\lambda} R(p')\pi^2\epsilon^2 + O(\epsilon^4 L^2) + O(\tfrac{\epsilon^2}{L^2}).
\end{aligned} \tag{4.10}$$

Together with (4.3)-(4.5) and (4.7)-(4.10), we obtain the following identity

$$\begin{aligned}
II_\epsilon(u_\epsilon) &= II_1 + II_2 + II_3 \\
&= -16\pi^2\log\lambda - 8\pi^2\log 8\pi^2 + \tfrac{8\pi^2}{3} - 16\pi^2 + 2\int_M QG - 16\pi^2 S_0 \\
&\quad -\tfrac{\epsilon^2\pi^2}{3\lambda^3}(\tilde{Q}(p')\sum_i(\tfrac{a_{ii}}{2} + 2a_i^2) + \textstyle\sum_i(a_ib_i + \tfrac{b_{ii}}{8}) - \tfrac{R(p')}{36}\tilde{Q}(p')) \\
&\quad + O(\tfrac{\epsilon^2}{L^2}) + O(\epsilon^{2+\alpha}) + O(\tfrac{1}{L^4}) + O(\epsilon^4L^2) + O((L\epsilon)^5).
\end{aligned} \tag{4.11}$$

Proof of Theorem 1.2 : we set $L = \frac{\log\frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}}}$, then

$$\epsilon^2 \gg O(\frac{\epsilon^2}{L^2}) + O(\epsilon^{2+\alpha}) + O(\frac{1}{L^4}) + O(\epsilon^4L^2) + O((L\epsilon)^5)$$

when ϵ is very small. Therefore, we get Theorem 1.2. □

5 The conformal case

In this section, we will discuss the local conformal flat case of Theorem 1.2.

In this situation, locally we may write

$$g = e^{2f}\sum_i dx^i \otimes dx^i \quad with \quad f = c_ix^i + \frac{1}{2}c_{ij}x^{ij} + O(r^3),$$

and

$$\tilde{Q} = \tilde{Q}(p') + b_i x^i + \frac{1}{2} b_{ij} x^{ij} + O(r^3).$$

Note that by the conformal property of P_g, the corresponding Green function have the following local expression:

$$G = -2\log|x| + S_0(p') + a_i x^i + \frac{1}{2} a_{ij} x^{ij} + O(r^3).$$

When $f = 0$, we can use Theorem 1.2 to obtain: if

$$\sum_i \left(\frac{a_{ii}}{2} + 2a_i^2 + \frac{1}{\tilde{Q}(p')}(a_i b_i + \frac{b_{ii}}{8})\right) > 0,$$

then (1.3) has a solution.

For the general case, we set $g' = e^{-2f} g$, then applying Lemma 3.5, we get $G'_{p'} = G + f$, and then

$$a'_i = a_i + c_i, \quad and \quad a'_{ii} = a_{ii} + c_{ii}.$$

Thus we have the following results

Theorem 5.1. *Let (M, g) be a close 4-dimensional manifold with $k = 8\pi^2$ and P_g is positive. Suppose further that it is locally conformal flat near p'. If*

$$\sum_i \frac{a_{ii} + c_{ii}}{2} + 2(a_i + c_i)^2 + \frac{1}{\tilde{Q}(p')}((a_i + c_i) b_i + \frac{b_{ii}}{8}) > 0,$$

then equation (1.3) has a minimal solution.

As a corollary, we have

Corollary 5.2. *With the same assumption as in Theorem 5.1. If*

$$\sum_i \frac{a_{ii} + c_{ii}}{2} + 2(a_i + c_i)^2 > 0,$$

then in the conformal class of (M, g) there is a constant Q-curvature.

To end this section, we propose the following conjecture:

Conjecture: *Let (M, g) be a locally conformal flat closed Riemannian manifold of dimension four, with $k = 8\pi^2$ and P_g is positive. Then we have*

$$\sum_i \left(\frac{a_{ii} + c_{ii}}{2} + 2(a_i + c_i)^2\right) \geq 0, \quad \textit{at the point } p' \text{ where } \Lambda_g(p') = \min_{x \in M} \Lambda_g(8\pi^2, x),$$

and the equality holds if and only if (M, g) is in the conformal class of the standard 4-sphere.

Let $\tilde{g} = e^{2G} g$, then we have

$$Q_{\tilde{g}}(x) = 0$$

for any $x \neq p$. Near p, we can write

$$\tilde{g} = \frac{e^{S_0(p)+(c_i+a_i)x^i+(c_{ij}+a_{ij})x^{ij}}}{r^2} = \frac{e^{S_0(p)}}{r^2}(\theta_i x^i + \theta_{ij}x^{ij} + O(|x|^3)).$$

So the above conjecture is equivalent to that

$$\sum_i \theta_{ii} > 0$$

when $M \neq S^4$. So, this problem is very similar to the positive mass problem.

6 Appendix

Suppose $KerP_g = \{constant\}$. Let G be the Green function which satisfies

$$P_g G + 2Q_g = 16\pi^2\delta_p.$$

As a corollary of a result in [N], we have the following

Lemma 6.1. *In a normal coordinate system of p, we have*

$$G = -2\log r + S_0 + a_i x^i + a_{ij}x^{ij} + O(r^{2+\alpha}).$$

However, for the reader's sake, we give a brief proof of this Lemma here:

Proof. In a normal coordinate system, we set

$$|g| = 1 - \frac{1}{3}R_{ij}x^{ij} + O(r^3), \quad and \quad g^{km} = \delta^{km} - \frac{1}{3}R_{kijm}x^{ij} + O(r^3)$$

where φ_{ijk} and θ_{ijk} are smooth.

Given a smooth function F, we have

$$\begin{aligned}
\Delta_g F(|x|) &= \tfrac{1}{\sqrt{|g|}}\tfrac{\partial}{\partial x^k}(\sqrt{|g|}g^{km}\tfrac{\partial}{\partial x^m}F) \\
&= \tfrac{\partial}{\partial x_k}(g^{km}F'\tfrac{x_m}{r}) + \tfrac{1}{2}g^{km}F_m\tfrac{\partial}{\partial x_k}\log|g| \\
&= \tfrac{\partial}{\partial x_k}(F'\tfrac{x_k}{r} - \tfrac{1}{3}R_{kijm}F'\tfrac{x^{kij}}{r} + F'O(r^3)) - \tfrac{1}{3}R_{ij}F'\tfrac{x^{ij}}{r} + O(F'r^2) \\
&= \tfrac{\partial}{\partial x_k}(F'\tfrac{x_k}{r} + F'O(r^3)) - \tfrac{1}{3}R_{ij}F'\tfrac{x^{ij}}{r} + O(F'r^2) \\
&= \Delta_0 F - \tfrac{1}{3}R_{ij}F'\tfrac{x^{ij}}{r} + O(F'r^2) + O(F''r^3).
\end{aligned}$$

Then

$$\Delta_g(-2\log r) = -\frac{4}{r^2} + \frac{2}{3}R_{ij}\frac{x^{ij}}{r^2} + O(r)$$

and

$$\Delta_g(-\frac{4}{r^2}) = \Delta_0(-\frac{4}{r^2}) - \frac{8R_{ij}x^{ij}}{3r^4} + O(\frac{1}{r}) = 16\pi^2\delta_0 - \frac{8R_{ij}x^{ij}}{3r^4} + O(\frac{1}{r}).$$

It is easy to check that

$$\Delta_g\frac{2}{3}R_{ij}\frac{x^{ij}}{r^2} = \Delta_0\frac{2}{3}R_{ij}\frac{x^{ij}}{r^2} + O(\frac{1}{r}) = \frac{4R}{3r^2} - \frac{16R_{ij}x^{ij}}{3r^4}.$$

Hence, we get

$$\Delta_g^2(-2\log r) = 16\pi^2\delta_p + \frac{4R}{3r^2} - 8\frac{R_{ij}x^{ij}}{r^4} + O(\frac{1}{r}).$$

Moreover, we have

$$\begin{aligned} div(\tfrac{2}{3}R_g(-d2\log r) - 2Ric_g\langle d(-2\log r), \cdot\rangle) &= \tfrac{2}{3}R_p(p')(2\log r)_{kk} - 2R_{km}(p')(2\log r)_{km} + O(\tfrac{1}{r}) \\ &= \tfrac{2}{3}R_g(p')\tfrac{4}{r^2} - 4R_g(p')\tfrac{1}{r^2} + 8R_{km}\tfrac{x^{km}}{r^4} + O(\tfrac{1}{r}). \end{aligned}$$

We therefore have

$$P_g(-2\log r) = 16\pi^2\delta_0 + O(\frac{1}{r}).$$

We set

$$G = -2\log r + S$$

where $S \in C^{1,\alpha}$. Then, we get

$$\Delta_g^2 S = P_g S + O(\frac{1}{r}) = P_g G + 2P_g \log r + O(\frac{1}{r}) = O(\frac{1}{r}).$$

This proves the lemma. □

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